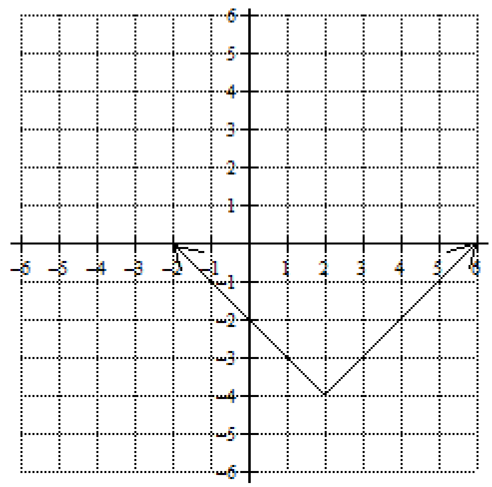
Lesson #3- Understanding the Limit
An Algebraic Approach

Consider the function, $f(x)=\frac{1}{2}\left|-2x+4\right|-4$, for a moment. The graph of $f(x)$ is pictured below. From the graph, determine the following limits.

$\lim_{x\to a} f(x)$	**Find $f(a)$ using the equation.**	**Find $\lim_{x\to a} f(x)$ from the graph.**
$\lim_{x\to 0} f(x)$		
$\lim_{x\to 2^+} f(x)$		
$\lim_{x\to 10} f(x)$		



When a function is defined and continuous at a value, $x = a$, how can $\lim_{x\to a} f(x)$ be found analytically?

Find each of the following limits analytically.

a) $\lim_{x\to 3}\frac{1}{2}x^2-2x+3$

b. $\lim_{x\to 3}\frac{5x+2}{2x-3}$

c. $\lim_{x\to 2^-}\frac{\sqrt{x+2}-1}{x+1}$

d. $\lim_{\theta\to \frac{\pi}{2}}\sin 2\theta$

e. $\lim_{\theta\to \frac{2\pi}{3}}\frac{\cos\theta}{\theta}$

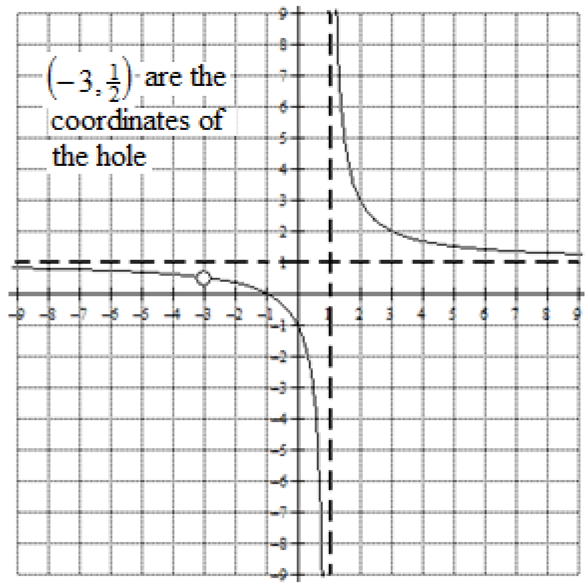
f. $\lim_{x\to 9}\log_8(11-x)$

Analytically Finding Limits of Functions at Undefined Values

What happens if we try to evaluate the limits below by the direct substitution method that was used in the previous six examples?

$$\lim_{x\to -3}\frac{x^2+4x+3}{x^2+2x-3} \qquad \lim_{x\to 1^-}\frac{x^2+4x+3}{x^2+2x-3} \qquad \lim_{x\to 1^+}\frac{x^2+4x+3}{x^2+2x-3}$$

Just because a function is undefined at a value of *x* does not mean that a conclusion cannot be reached about the limit. Consider the rational function above. From the graph of the function pictured to the right, what is the value of each limit below?



$$\lim_{x\to -3}\frac{x^2+4x+3}{x^2+2x-3} = \_\_\_\_\_\_\_\_$$

$$\lim_{x\to 1^-}\frac{x^2+4x+3}{x^2+2x-3} = \_\_\_\_\_\_\_\_$$

$$\lim_{x\to 1^+}\frac{x^2+4x+3}{x^2+2x-3} = \_\_\_\_\_\_\_\_$$

The task now is to determine how to find these limits analytically. How was it that we found the discontinuities of a rational function in pre-calculus?

We will perform the same algebraic analysis to find the limit of the removable, point discontinuities. Let's do this **Cancellation Process** below.

$$\lim_{x\to -3}\frac{x^2+4x+3}{x^2+2x-3}$$

Based on our knowledge from pre-calculus, we know that if a rational function has a non-removable infinite discontinuity, graphically a ______________________________ exists. Since the y – values do not approach one specific value from both sides at a ____________________________, then the limit does not exist. However, we can determine if the one sided limits approach $-\infty$ or ∞. In order to do this analytically, we will marry the numerical, graphical, and algebraic approaches. For each limit below, determine the sign of the simplified function at the value to the right or the left of $x = 1$.

$$\lim_{x\to 1^-} \frac{x^2+4x+3}{x^2+2x-3}$$

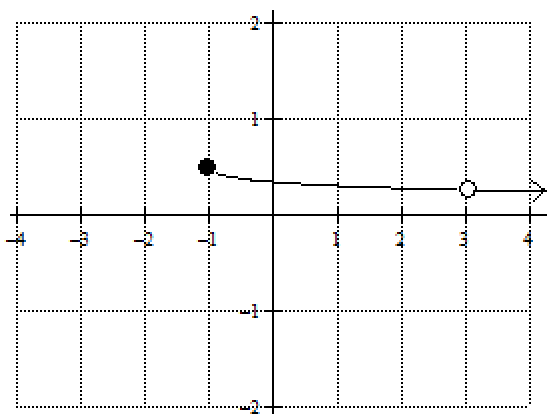
Value of x to the left of $x = 1$	Simplified function $\frac{x+1}{x-1}$
0.9	

$$\lim_{x\to 1^+} \frac{x^2+4x+3}{x^2+2x-3}$$

Value of x to the right of $x = 1$	Simplified function $\frac{x+1}{x-1}$
1.1	

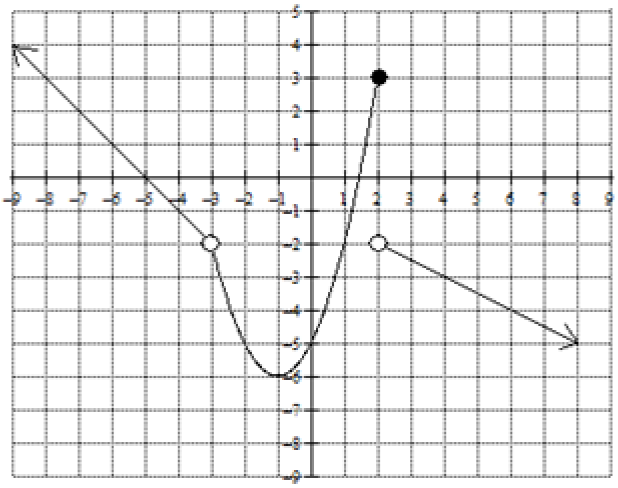
The graph of a function $g(x) = \dfrac{\sqrt{x+1}-2}{x-3}$ is pictured to the right. Often, **<u>rationalization</u>** can be used to evaluate a limit analytically. Find the following limit.

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$$



Write the equation of the piece-wise defined function pictured to the right.

	Equation of Each Piece		Constraint of Each Piece
		,	
$f(x) =$		,	
		,	



Use the equation that you just wrote to find each of the following limits. Confirm your results based on the graph. If a limit does not exist, state why.

a) $\lim_{x \to 2^+} f(x)$

b) $\lim_{x \to 2} f(x)$

c) $\lim_{x \to -3^-} f(x)$

d) $\lim_{x \to -3^+} f(x)$

e) $\lim_{x \to -7} f(x)$

f) $\lim_{x \to -1} f(x)$

Consider the function below to find each limit. If a limit does not exist, state why.

$$G(x)=\begin{cases}2x^2+3x, & x<-2\\ -\frac{1}{2}x+1, & x>-2\end{cases}$$

a) $\lim\limits_{x\to -2^-} G(x)$

b) $\lim\limits_{x\to -2^+} G(x)$

c) $\lim\limits_{x\to -2} G(x)$

Find each of the following limits analytically. Show your algebraic analysis.

a. $\lim\limits_{x\to e} \dfrac{\ln x}{2x}$
b. $\lim\limits_{x\to 5^-} \left(\frac{2}{5}x^2+2x\right)$
c. $\lim\limits_{\theta\to \pi} \left(\sin^2\theta+2\cos\theta\right)$
d. $\lim\limits_{\alpha\to \frac{5\pi}{3}} \dfrac{\tan\alpha}{\alpha^2}$
e. $\lim\limits_{x\to -2} \dfrac{x^2-x-6}{2x+4}$

f. $\lim_{x \to 3} \frac{x+5}{x^2-9}$

g. $\lim_{x \to \frac{3}{2}} \frac{8x^3-27}{2x-3}$

h. $\lim_{x \to -2} \frac{\sqrt{2x+5}-1}{x+2}$

i. $\lim_{x \to 1} \frac{1-\sqrt{2x^2-1}}{x-1}$

j. $\lim_{x\to 0} \frac{\frac{1}{x+2}+\frac{1}{x}}{x}$

k. $\lim_{x\to 2^+} \frac{3x^2+7x+2}{x^2-4}$

l. $\lim_{x\to 3^+} \frac{2x+5}{x-3}$

m. $\lim_{x\to 3^-} \frac{2x+5}{x-3}$

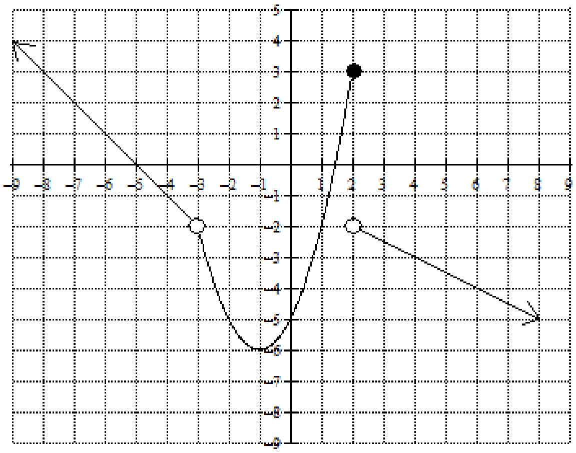
If $f(x) = 2x^2 - 3x + 4$, find $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

Properties of Limits

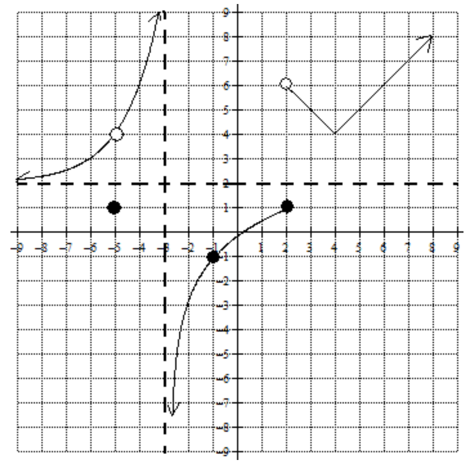
Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Find each of the following limits in terms of *L* and *M*.

1.	$\lim_{x\to a}\left[f(x)+g(x)\right]$
2.	$\lim_{x\to a}\left[f(x)-g(x)\right]$
3.	$\lim_{x\to a}\frac{f(x)}{g(x)}$
4.	$\lim_{x\to a}\left[f(x)\cdot g(x)\right]$
5.	$\lim_{x\to a}\left[c\cdot f(x)\right]$
6.	$\lim_{x\to a}\left[f(x)\right]^p$
7.	$\lim_{x\to a} c$

Graph of $f(x)$



Graph of $g(x)$



Find each of the following limits applying the properties of limits. If a limit does not exist, state why.

$\lim_{x\to 2^-}\left[f(x)+g(x)\right]$	$\lim_{x\to -1}\left[2f(x)-3g(x)\right]$	$\lim_{x\to -3}\left[f(x)-g(x)\right]$
$\lim_{x\to 6}\frac{-2f(x)}{g(x)}$	$\lim_{x\to 4}2\left[f(x)g(x)\right]$	$\lim_{x\to -2}\left[f(x)\right]^2$
$\lim_{x\to 2^+}\sqrt{2g(x)}$	$\lim_{x\to 2^+}\frac{f(x)}{g(x)}$	$\lim_{x\to -1}\left[f(x)-g(x)\right]$

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Lesson #3 Homework

Find the value of each limit. For a limit that does not exist, state why.

1. $\lim\limits_{x \to -\frac{1}{2}} 3x^2(2x-1)$	2. $\lim\limits_{x \to -1} x^3 + 2x^2 - 3x + 3$
3. $\lim\limits_{x \to -2} (x-6)^{2/3}$	4. $\lim\limits_{x \to 2} \frac{x^2+5x+6}{x+2}$
5. $\lim\limits_{\theta \to \frac{\pi}{6}} \theta^2 \tan\theta$	6. $\lim\limits_{x \to 0} \frac{(x+4)^2-16}{x}$
7. $\lim\limits_{x \to 1} \frac{x-1}{x^2-1}$	8. $\lim\limits_{x \to 2} \frac{x^2-3x+2}{x^2-4}$

9. $\lim_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2}$	10. $\lim_{x\to 0} \frac{\frac{1}{x+2}-\frac{1}{2}}{x}$
11. $\lim_{x\to 0} \frac{(2+x)^3-8}{x}$	12. $\lim_{h\to 0} \frac{(x+h)^2+2(x+h)-3-(x^2+2x-3)}{h}$
13. $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ if $f(x) = 3x^2 - 2x$	14. $\lim_{x\to 2} f(x)$ if $f(x) = \begin{cases} 2x^2-4x, & x<2 \\ 4\sin\left(\frac{\pi x}{4}\right), & x>2 \end{cases}$

15. $\lim_{x \to 3} e^x \cos\left(\frac{\pi x}{3}\right)$	16. $\lim_{x \to 1} \frac{\sqrt{x+3}-2}{x-1}$
17. $\lim_{x \to 3^+} \frac{x+3}{x-3}$	18. $\lim_{x \to -3^+} \frac{2x^2-9x+9}{x^2-9}$
19. $\lim_{x \to 0} \frac{\frac{1}{x-2}+\frac{1}{2}}{x}$	20. $\lim_{x \to -2} \begin{cases} 2-x, & x < -2 \\ x^2-2x, & x > -2 \end{cases}$

21. If $\lim_{x \to 3} f(x) = 2$ and $\lim_{x \to 3} g(x) = -4$, find each of the following limits. Show your analysis applying the properties of limits.

a. $\lim_{x \to 3}\left[\frac{5f(x)}{g(x)}\right]$	b. $\lim_{x \to 3}\left[f(x)+2g(x)\right]$	c. $\lim_{x \to 3}\sqrt{4f(x)}$
d. $\lim_{x \to 3}\frac{g(x)}{8}$	e. $\lim_{x \to 3}\left[3f(x)-g(x)\right]$	f. $\lim_{x \to 3}\left[\frac{f(x)g(x)}{12}\right]$

22. If $\lim_{x \to 4} f(x) = 0$ and $\lim_{x \to 4} g(x) = 3$, find each of the following limits. Show your analysis applying the properties of limits.

a. $\lim_{x \to 4}\left[\frac{g(x)}{f(x)-1}\right]$	b. $\lim_{x \to 4} xf(x)$
c. $\lim_{x \to 4}\left[g(x)+3\right]$	d. $\lim_{x \to 4} g^2(x)$