Lesson #5- Limits of Trigonometric Functions
An Analytical Approach

We have already looked at how to evaluate limits of trigonometric functions by direct substitution, provided that the function is defined and continuous at θ. Find each of the limits below.

$$\lim_{\theta \to \frac{2\pi}{3}} \frac{\sin 3\theta}{3\theta}$$	$$\lim_{\theta \to \pi} 2\cos^2 \theta$$

Each of the functions above was defined at the value that θ was approaching. However, we have seen that even in the algebraic world, not all functions are undefined at a value, but their limits do exist. The same is true in the trigonometric world.

Evaluating Trigonometric Limits by Rewriting the Function Using Identities

Let's consider for a moment the limit below. Try to evaluate this limit by direct substitution.

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\sin^2 \theta}$$

Again, this function is undefined at $\theta = 0$. However, that does not mean that the limit does not exist. In this case, we can often rewrite the function in terms of a single trig ratio using identities in hopes that the new form of the function is not undefined for the approached value of θ. Do this in the space below.

Find each of the following limits by rewriting the function in a form that is defined for the approached value of θ.

1. $\displaystyle\lim_{\theta\to\frac{3\pi}{2}} 3\tan\theta\cos\theta$	2. $\displaystyle\lim_{\theta\to\frac{\pi}{2}} \frac{\sec\theta\cos\theta}{4\theta}$	3. $\displaystyle\lim_{\theta\to\pi} \frac{\cos\theta\tan\theta}{\sin\theta}$

It is important to note that as in all cases of evaluating limits, direct substitution should always be tried FIRST. If that does not yield a value, then a simplification of the function can be tried.

Occasionally, it is not even possible to rewrite the function so that it is undefined. There are two special trigonometric limits that can often be employed.

Use your graphing calculator to complete the table of values below for the function $f(\theta) = \dfrac{\sin\theta}{\theta}$.

θ	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$\dfrac{\sin\theta}{\theta}$						

Based on the values in the table above, what do each of the limits below equal?

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = \underline{\quad\quad} \qquad \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = \underline{\quad\quad} \qquad \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \underline{\quad\quad}$$

Use your graphing calculator to complete the table of values below for the function $f(\theta) = \dfrac{\sin 3\theta}{3\theta}$.

θ	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$\dfrac{\sin 3\theta}{3\theta}$						

Based on the values in the table above, what do each of the limits below equal?

$$\lim_{\theta \to 0^-} \frac{\sin 3\theta}{3\theta} = \underline{\quad\quad} \qquad \lim_{\theta \to 0^+} \frac{\sin 3\theta}{3\theta} = \underline{\quad\quad} \qquad \lim_{\theta \to 0} \frac{\sin 3\theta}{3\theta} = \underline{\quad\quad}$$

Based on what you have just observed, what inference can you make about the value of the limit $\lim_{\theta \to 0} \dfrac{\sin c\theta}{c\theta}$, where c is any constant?

Now, in a similar fashion, use your graphing calculator to complete the table of values below for the function $f(\theta) = \dfrac{1 - \cos\theta}{\theta}$.

θ	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$\dfrac{1 - \cos\theta}{\theta}$						

Based on the values in the table above, what do each of the limits below equal?

$$\lim_{\theta \to 0^-} \frac{1 - \cos\theta}{\theta} = \underline{\quad\quad} \qquad \lim_{\theta \to 0^+} \frac{1 - \cos\theta}{\theta} = \underline{\quad\quad} \qquad \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = \underline{\quad\quad}$$

These two special trigonometric functions derived above can often be used to find limits of trigonometric functions that cannot be evaluated by direct substitution nor by rewriting the function using identities.

Find each of the following limits.

1. $\displaystyle\lim_{x\to 0}\frac{e^x\cos x}{4}$	2. $\displaystyle\lim_{\theta\to 0}\frac{\sin 4\theta}{\theta}$
3. $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{3x}$	4. $\displaystyle\lim_{\theta\to 0}\frac{2\sin 5\theta}{3\theta}$
5. $\displaystyle\lim_{\theta\to 0}\frac{\tan\theta}{\theta}$	

6. $\displaystyle\lim_{\theta \to 0} \frac{2 - 2\cos^2 \theta}{\theta}$

7. $\displaystyle\lim_{\theta \to 0} \frac{1 - \cos\theta + \sin 2\theta}{\theta}$

8. $\displaystyle\lim_{\theta \to 0} \frac{\theta\csc\theta + 1}{\theta\csc\theta}$

9. $\displaystyle\lim_{x \to 0} \frac{\sin x - \sin x \cos x}{x^2}$

Lesson #5 Homework

Find the value of each limit. For a limit that does not exist, state why.

1. $\lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{\cos^2 \theta}{1-\sin \theta}$	2. $\lim\limits_{x \to 0} \dfrac{x+\sin x}{x}$
3. $\lim\limits_{x \to 3} \begin{cases} 2x^2 - 3x, & x < 3 \\ 8 - \cos\left(\frac{\pi x}{3}\right), & x > 3 \end{cases}$	4. $\lim\limits_{\theta \to 0} \dfrac{2\sin 3\theta}{\theta}$

5. $\displaystyle\lim_{x\to 0}\frac{\sin x}{2x^2-x}$

6. $\displaystyle\lim_{x\to 0}\frac{5x+\sin 3x}{x}$

7. $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{6x}$

8. $\displaystyle\lim_{x\to 0}\frac{2\sin 4x}{3x}$

9. $\displaystyle\lim_{\theta\to 0}\frac{\cos\theta\tan\theta}{3\theta}$

10. $\displaystyle\lim_{\theta\to 0}\frac{3-3\cos\theta}{\theta}$

11. $\displaystyle\lim_{\theta\to\frac{\pi}{2}}\frac{\cos\theta}{\cot\theta}$

12. $\displaystyle\lim_{\theta\to 0}\frac{1-\tan\theta}{\sin\theta-\cos\theta}$

13. $\displaystyle\lim_{c\to 3}\frac{c^3-27}{c-3}$

14. $\displaystyle\lim_{x\to -1}\frac{(x+3)^3-8}{x+1}$