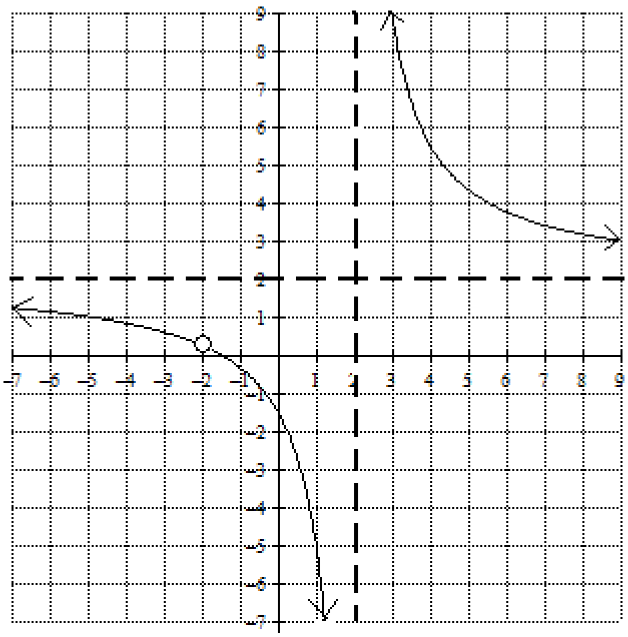
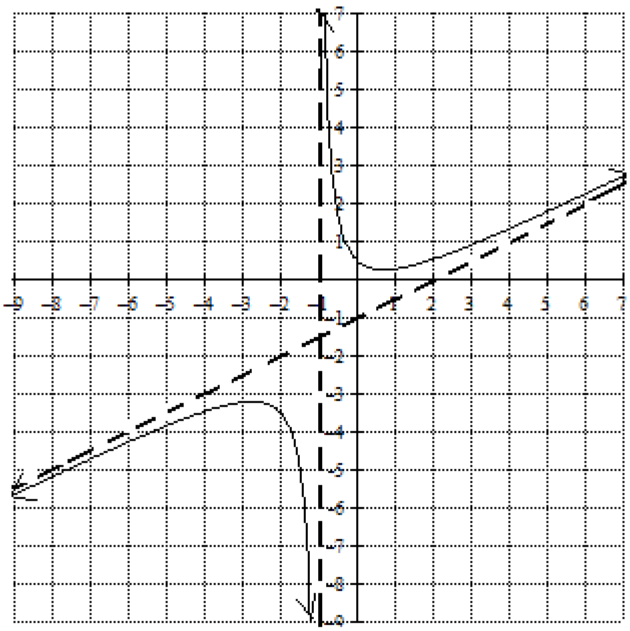
Lesson #8- Infinite Limits and Limits at Infinity

In our graphical analysis of limits, we have already seen both an infinite limit and a limit at infinity. Let's consider the equations and the graphs of the two functions below to find the limits that follow.

$$f(x)=\frac{2x^2+7x+6}{x^2-4} \qquad g(x)=\frac{x^2-x+1}{2x+2}$$





Infinite Limits	**Limits at Infinity**
$\lim\limits_{x\to 2^-} f(x)=$	$\lim\limits_{x\to\infty} f(x)=$
$\lim\limits_{x\to 2^+} f(x)=$	$\lim\limits_{x\to-\infty} f(x)=$
$\lim\limits_{x\to -1^-} g(x)=$	$\lim\limits_{x\to\infty} g(x)=$
$\lim\limits_{x\to -1^+} g(x)=$	$\lim\limits_{x\to-\infty} g(x)=$

We have already seen how to find infinite limits by marrying a numerical and analytical approach. For the function, $f(x)$ and $g(x)$, whose graphs appear on the previous page, find the infinite limits below.

$\lim\limits_{x \to 2^-} \dfrac{2x^2+7x+6}{x^2-4}$	$\lim\limits_{x \to 2^+} \dfrac{2x^2+7x+6}{x^2-4}$
$\lim\limits_{x \to -1^+} \dfrac{x^2-x+1}{2x+2}$	$\lim\limits_{x \to -1^-} \dfrac{x^2-x+1}{2x+2}$

Graphically, an infinite limit will always yield a ____.

In pre-calculus, we discovered through observation that such a graphical property existed when a factor in the equation would not ____. From this point forward, this is NOT a viable justification for the existence of a ___________________________________.

Justification of the Existence of a Vertical Asymptote Using Limits

For the function below, find any vertical asymptote(s) that exist. Justify your answer(s) using a limit(s).

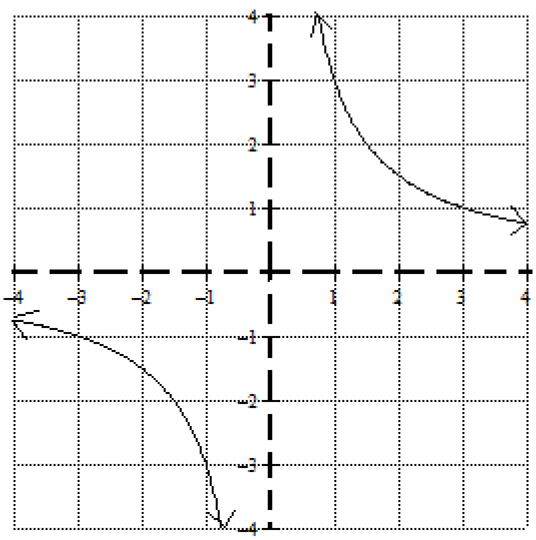
$$h(x)=\frac{2x^2+7x+3}{x^2+2x-3}$$

Now, we must develop an analytical procedure by which we can find limits at infinity. Basically, a limit at infinity describes the end behavior of a function. We have spent a great amount of time talking about end behavior of functions. Find each of the following limits at infinity. Give an explanation of your reasoning for each.

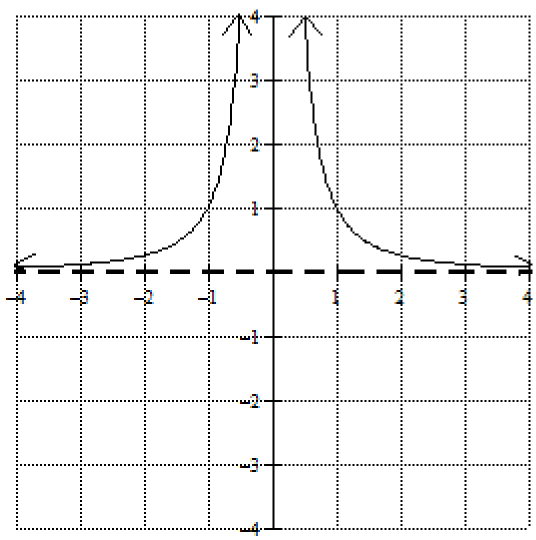
$\lim\limits_{x\to\infty} -3x^3 + x - 4$	$\lim\limits_{x\to-\infty} (4-x)^2(x-3)(x+1)$	$\lim\limits_{x\to\infty} \dfrac{3-2x}{x+3}$

The third example, $\lim\limits_{x\to\infty} \dfrac{3-2x}{x+3}$ will provide us a basis for developing our analytical process by which we can find limits at infinity for all types of rational functions. Before we do that, investigate the two functions below both graphically and numerically.

$$f(x) = \frac{3}{x}$$



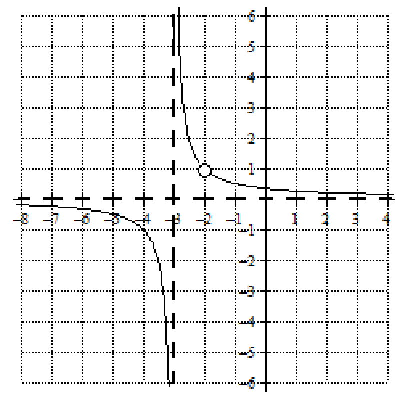
$$g(x) = \frac{1}{x^2}$$



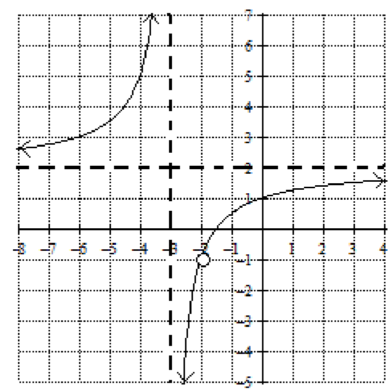
What does each of these functions have in common algebraically and what do they have in common graphically?

In pre-calculus, we learned three rules for determining the existence of horizontal asymptotes of rational functions. When a rational function had a horizontal asymptote, the end behavior was always such that as $x \to -\infty$ or ∞, then the graph of $f(x) \to$ the horizontal asymptote. We learned three rules for determining the horizontal asymptote, if one existed, for rational functions. We are about to use the idea of a limit and calculus to find out why those rules are such as they are. For each function below, divide every term in both the numerator and the denominator by the highest power of x that appears in the denominator. Then, evaluate the indicated limit. Does the result of each limit make sense based on the graph that is pictured?

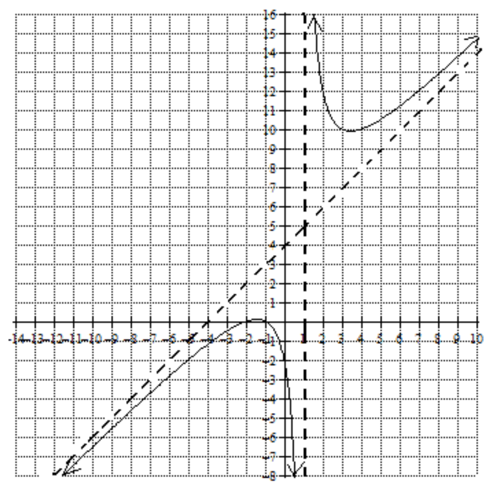
$$\lim_{x\to\infty} \frac{x+2}{x^2+5x+6}$$



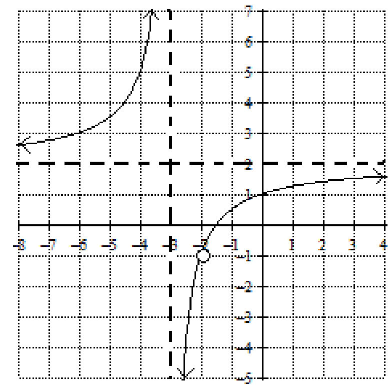
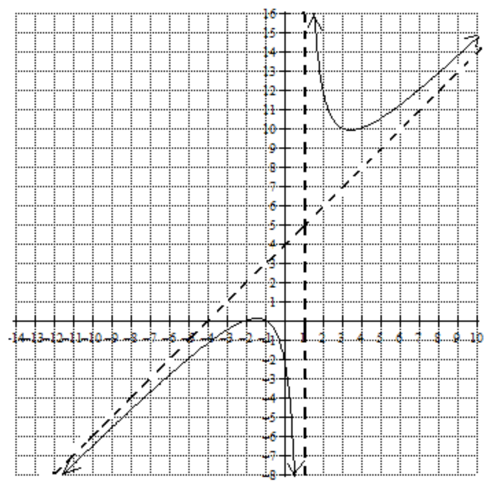
$$\lim_{x\to\infty} \frac{2x^2+7x+6}{x^2+5x+6}$$



$$\lim_{x\to\infty} \frac{x^2+3x+2}{x-1}$$



Let’s see what would happen if our limit at infinity approached $-\infty$.

$\lim\limits_{x\to-\infty} \dfrac{2x^2+7x+6}{x^2+5x+6}$	
$\lim\limits_{x\to-\infty} \dfrac{x^2+3x+2}{x-1}$	

Based on what we have just seen and what we know graphically about the functions above, would does approaching $-\infty$ make a difference in the result of our limits?

Graphically, a limit at infinity can yield a ____. In pre-calculus, we discovered through observation that such a graphical property existed by comparing the

____.

From this point forward, this is NOT a viable justification for the existence of a

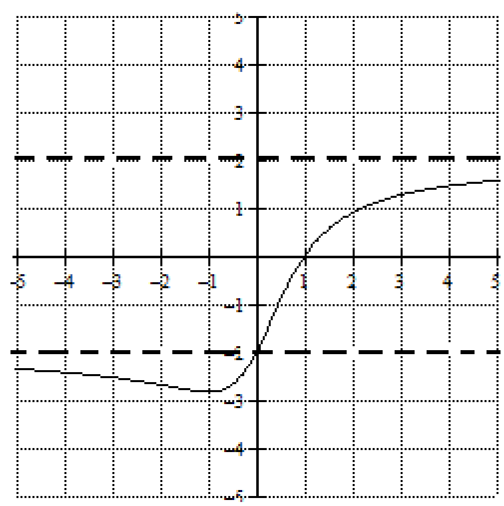
__________________________________.

Justification of the Existence of a Horizontal Asymptote Using Limits

For the function below, find the horizontal asymptote if it exists. Justify your answer(s) using a limit(s).

$$h(x)=\frac{5-2x-2x^2}{3x^2+2x-3}$$

The algebraic analysis described above to evaluate a limit at infinity can be used to find limits at infinity for any type of rational function, even $f(x)=\dfrac{2x-2}{\sqrt{x^2+1}}$, whose graph is pictured to the right.



What is the one thing that you notice is different about the graph of this rational function versus the others that we have investigated in the past?

Use the graph to find each of the following limits.

$$\lim_{x\to-\infty}\frac{2x-2}{\sqrt{x^2+1}}=\_\_\_\_\_\_\_\_\_\_\_\_\qquad \lim_{x\to\infty}\frac{2x-2}{\sqrt{x^2+1}}=\_\_\_\_\_\_\_\_\_\_\_\_$$

Perform the same algebraic analysis that we did earlier to find the limits at infinity. The only problem that we will encounter is what to do when $x \to -\infty$.

$\lim\limits_{x\to\infty} \frac{2x-2}{\sqrt{x^2+1}}$	$\lim\limits_{x\to-\infty} \frac{2x-2}{\sqrt{x^2+1}}$

Look at the graph on the previous page to confirm these results. Then, find the limits below.

$\lim\limits_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}}$	$\lim\limits_{x\to-\infty} \frac{x^2+2x}{\sqrt{3x^2+2}}$

Name____**Date**____________________**Class**__________

Lesson #8 Homework

For exercises 1 – 3, determine the vertical asymptotes of each function. Justify your answers by using limits, showing your numerical analysis.

1. $g(x)=\dfrac{x^2-x-6}{x+5}$	2. $f(x)=\dfrac{x-5}{x^2+x-2}$	3. $h(x)=\dfrac{2x^2+x-3}{x^2-3x+2}$

Find each of the following limits at infinity. What do the results show about the existence of a horizontal asymptote? Justify your reasoning.

4. $\lim\limits_{x\to-\infty}\dfrac{3x+2-5x^2}{2x^2-3x-1}$	5. $\lim\limits_{x\to\infty}\dfrac{3x+5}{2x^2-3x}$	6. $\lim\limits_{x\to-\infty}\dfrac{-2x^2+5}{3x+2}$

Find each of the following limits at infinity. What do the results show about the existence of a horizontal asymptote? Justify your reasoning.

7. $\lim\limits_{x \to -\infty} \dfrac{2x+1}{\sqrt{x^2 - x}}$	8. $\lim\limits_{x \to \infty} \dfrac{-2x^2 + x}{\sqrt{2x^2 - 3}}$